

ON THE REALIZABILITY OF GROUP ACTIONS

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ABSTRACT. We raise the question of realizability of group actions which is an extended version of the 1960's Kahn realizability problem for (abstract) groups. Namely, if M is a $\mathbb{Z}G$ -module for a group G , we say that a simply-connected space X realize this action if, for some k , $\pi_k(X)$ as a $\mathbb{Z}\mathcal{E}(X)$ -module for the group $\mathcal{E}(X)$ of self-homotopy equivalences of X , is isomorphic to M as a $\mathbb{Z}G$ -module. Which modules can be so realized? In this paper we obtain a positive answer for any faithful finitely generated $\mathbb{Q}G$ -module, where G is finite. Our proof relies on providing a positive answer to Kahn's problem for a large class of orthogonal groups of which, by using invariant theory, our case is shown to be a particular one.

1. INTRODUCTION

Realizability problems in algebraic topology are very easy to state and extremely difficult to solve. Classical examples of this kind are the *realizability of cohomological algebras* settled by N. E. Steenrod in the 1960's, which asks for characterization of graded algebras that appear as the cohomology algebra of a space, [1], [2]; the *G -Moore space problem*, also by Steenrod, which asks for characterization of $\mathbb{Z}G$ -modules that appear as the homology of some simply-connected G -Moore space X , [15, Problem 51], [6]; the *realizability problem for abstract groups* proposed by D. Kahn, which asks for characterization of groups that appear as the group of self-homotopy equivalences of a simply-connected spaces, [13], [8]. They all ask which algebraic structures occur as a given homotopy-type of a space.

In this paper we present a question which is an extension, and an homotopic dual, respectively, of the Kahn realizability problem and the G -Moore space problem presented above. To be more precise, we say that a $\mathbb{Z}G$ -module M is realized by a simply-connected space X if it occurs as the $\mathbb{Z}\mathcal{E}(X)$ -module $\pi_k(X)$ for some k , where $\mathcal{E}(X)$ is the group of self-homotopy equivalences with its natural action (by composition) on the homotopy groups $\pi_k(X)$, for every $k \geq 2$.

Then, one might ask the following question:

Question 1.1 (Realizability of group actions). *Which finitely generated $\mathbb{Z}G$ -modules can be realized by simply-connected spaces X ?*

Restrictions on this question are studied here, mainly because realizing a group action implies in particular to realize (in the Kahn sense) the group itself, which is still an open problem in the general case. Key progress was made though in a recent work [8], where

First author is partially supported by Ministerio de Economía y Competitividad grant MTM2013-43687-P (European FEDER support included). Second author is partially supported by Ministerio de Economía y Competitividad grant MTM2013-41768-P (European FEDER support included). Both authors are partially supported by Xunta de Galicia grant EM 2013/016.

we proved that every finite group is the group of self homotopy equivalences of a simply-connected rational space. The use of rational homotopy theory techniques and the ingenious result of Frucht [11] that expresses every finite group as the automorphism group of a finite graph, allowed us to realize any finite group through a minimal Sullivan algebra of finite type (a rational space) encoding the *finite* set of vertices and edges of a graph. Here, we exploit the same techniques of rational homotopy theory, though we follow a significantly different approach that will be carried out in Section 2. Namely, if the action is faithful over a finitely generated $\mathbb{Q}G$ -module M , and G is finite, we make a connection with Invariant Theory to represent $G \leq \mathrm{GL}(M)$ as $\mathrm{O}(\mathcal{Q}) \leq \mathrm{GL}(M)$, the inclusion of the orthogonal group of a family \mathcal{Q} of algebraic forms over M in the general linear group, Theorem 2.3. Then, in Section 3, we show that a large class of orthogonal groups $\mathrm{O}(\mathcal{Q})$ can be realized in the Kahn sense by a minimal Sullivan algebra encoding the algebraic forms, Theorem 3.2 and Corollary 3.3. The general result that we prove here and that we obtain as an application of Theorem 3.2 and Theorem 2.3 above mentioned is as follows.

Theorem 1.2. *Every faithful and finitely generated $\mathbb{Q}G$ -module M where G is a finite group, can be realized by infinitely many (non-homotopy equivalent) rational spaces X .*

In the previous theorem, spaces X are Postnikov pieces, or in other words, G -objects in the homotopy category of topological spaces and, the module M appears as $\pi_{40}(X)$ whereas the G -action on $\pi_{\neq 40}(X)$ is trivial. In that sense, Question 1.1 is not only a generalization of Kahn's problem, but an homotopic dual of the G -Moore problem aforementioned. Remark that this is the best one can do as a dual, since if X was an Eilenberg-MacLane space, namely $X = K(M, n)$, then G would be isomorphic to $\mathrm{Aut}(M)$ which is an infinite group for $M \neq \{0\}$ and would contradict our hypothesis.

The present work represents a further significant progress with respect to [8] since the class of orthogonal groups for which Theorem 3.2 is applied to, strictly contains finite groups. This latter approach leads to an alternate way of tackling Kahn's realizability problem. Section 4, devoted to that end, should be thought of as a first attempt to give a complete answer to the question of whether any group is the group of self homotopy equivalences of a rational space.

Henceforth M denotes an n -dimensional \mathbb{Q} -module.

2. FINITE GROUPS AS ORTHOGONAL GROUPS OF ALGEBRAIC FORMS

We use in this section Invariant Theory and we recall some pertinent facts, referring to [4] for more information. The ring of polynomial functions on M is denoted by $\mathbb{Q}[M]$, which by fixing a basis $\{v_1, \dots, v_n\}$ of the dual M^* , can be expressed as $\mathbb{Q}[v_1, \dots, v_n]$. An *n -algebraic form of degree d* is an homogeneous polynomial function $q(v_1, \dots, v_n)$ of degree d . We say that $f \in \mathrm{GL}(M)$ is an *automorphism of the n -algebraic form q* if $q \circ f = q$. The set of all the automorphisms of the form q is a group which we call *the orthogonal group of q* and we denote it by $\mathrm{O}(q)$. In an analogous way, we define the *orthogonal group of a family of n -algebraic forms $\mathcal{Q} = \{q_0, \dots, q_r\}$* by $\mathrm{O}(\mathcal{Q}) = \{f \in \mathrm{GL}(M) \mid q_i \circ f = q_i \text{ for every } i = 0, \dots, r\}$.

If a group G acts on M , the action of G over M extends to an action over $\mathbb{Q}[M]$ via $(g \cdot p)(u) = p(g^{-1}u)$ for $g \in G$, $p \in \mathbb{Q}[M]$. The invariant ring $\mathbb{Q}[M]^G$ is the set of all the fixed

points of this action which is the basic object of study of Invariant Theory. A classical result, due to D. Hilbert and E. Noether, which is central to this paper, states that if G is a finite group then the invariant ring is finitely generated; see [4, Corollary 1.3.2]. One can also look at the action of G over the field of fractions $\mathbb{Q}(M)$ given by $g \cdot (f_1/f_2) = (g \cdot f_1)/(g \cdot f_2)$. In this case, $\mathbb{Q}(M)$ is a Galois extension of the fixed points of this action, $\mathbb{Q}(M)^G$, with Galois group G , and $\mathbb{Q}(M)^G$ is the field of fractions of the invariant ring; see [4, Proposition 1.1.1].

Before proving a technical lemma, we introduce some notation.

Definition 2.1. *A family of n -algebraic forms $\mathcal{Q} = \{q_0, \dots, q_{r+1}\} \subset \mathbb{Q}[v_1, \dots, v_n]$ is realizable if:*

- (1) $q_{r+1} = q_0^s$ for some $s \geq \max\{n, \lceil \frac{\deg(q_r)}{\deg(q_0)} \rceil + 1\}$
- (2) $\deg(q_{i+1}) - \deg(q_i) > 1$, for $i = 0, \dots, r$.
- (3) $q_0 = \lambda_1 v_1^d + \dots + \lambda_n v_n^d$, for some integer $d > 1$, $\lambda_i \neq 0$ for $i = 1, \dots, n$.

We say that \mathcal{Q} is pre-realizable if \mathcal{Q} satisfies (1) and (2).

Now, recall that the ring of polynomial functions is a unique factorization domain. Then, we can prove the following lemma.

Lemma 2.2. *Let $\mathcal{P} = \{p_0, \dots, p_r\}$ be an arbitrary family of n -algebraic forms on M . Then, there exists a pre-realizable family $\mathcal{Q} = \{q_0, \dots, q_{r+1}\}$ of n -algebraic forms on M such that $\mathcal{O}(\mathcal{P}) = \mathcal{O}(\mathcal{Q})$ with $q_0 = p_0$.*

Proof. Let q_0 equal p_0 . Inductively, we define $q_i = p_i q_{i-1} q_0$ for $i = 1, \dots, r$ and $q_{r+1} = q_0^s$ for $s \geq \max\{n, \lceil \frac{\deg(q_r)}{\deg(q_0)} \rceil + 1\}$. It is immediate that (1) and (2) from Definition 2.1 hold for the family $\mathcal{Q} = \{q_0, \dots, q_{r+1}\}$, so it only remains to prove that the orthogonal groups coincide. Now, by induction, if $g \in \mathcal{O}(\mathcal{P})$, then $p_i \circ g = p_i$ for all i , and therefore

$$\begin{aligned} q_i \circ g &= (p_i q_{i-1} q_0) \circ g = (p_i \circ g)(q_{i-1} \circ g)(q_0 \circ g) = p_i q_{i-1} q_0 \\ &= q_i, \text{ for } i = 1, \dots, r, \text{ and} \\ q_{r+1} \circ g &= q_0^s \circ g = (q_0 \circ g)^s = q_0^s \\ &= q_{r+1}, \end{aligned}$$

hence $g \in \mathcal{O}(\mathcal{Q})$. On the other hand, if $g \in \mathcal{O}(\mathcal{Q})$ then

$$\begin{aligned} p_i q_{i-1} q_0 &= q_i \\ &= q_i \circ g \\ &= (p_i q_{i-1} q_0) \circ g = (p_i \circ g)(q_{i-1} \circ g)(q_0 \circ g), \text{ for } i = 1, \dots, r, \\ &= (p_i \circ g) q_{i-1} q_0, \text{ for } i = 1, \dots, r. \end{aligned}$$

Since $\mathbb{Q}[M]$ a unique factorization domain, then $p_i = p_i \circ g$, so $g \in \mathcal{O}(\mathcal{P})$. □

With these results in mind, we can prove the following result.

Theorem 2.3. *Let M be a faithful n -dimensional $\mathbb{Q}G$ -module where G is a finite group. Then, there exists a family $\mathcal{Q} = \{q_0, \dots, q_{r+1}\}$ of realizable n -algebraic forms over M such that $\deg(q_0) = 2$ and G is the orthogonal group $\mathcal{O}(\mathcal{Q}) \leq \text{GL}(M)$.*

Proof. We can consider $G \leq \mathrm{GL}(M)$. Now, as the invariant ring $\mathbb{Q}[M]^G$ is finitely generated, there exists a family of n -algebraic forms $\mathcal{P} = \{p_1, \dots, p_r\}$ such that if $g \in G$ then $g \cdot p_i = p_i$ for every $i = 1, \dots, r$. Now, take any $g \in \mathrm{GL}(M)$ such that $g \cdot p_i = p_i$ for every $i = 1, \dots, r$. In particular, g fixes every element in the field of fractions of the invariant ring, $\mathbb{Q}(M)^G$. Then, as $G = \mathrm{Gal}(\mathbb{Q}(M)/\mathbb{Q}(M)^G)$ we conclude that $g \in G$. So, we have proved that any element $g \in \mathrm{GL}(M)$ that fixes every n -algebraic form in \mathcal{P} , is in fact in G . In other words, we obtain that $G = \mathrm{O}(\mathcal{P})$.

Consider now the following positive definite form $p_0 = w_1^2 + \dots + w_n^2$, for an arbitrary basis of M^* , and transform it into a G -invariant positive definite n -form as follows: $q_0 = \sum_{g \in G} g \cdot p_0$.

Since every n -form of degree 2 over \mathbb{Q} can be diagonalized, for a basis $\{v_1, \dots, v_n\} \subset M^*$ that we fix, we can write $q_0 = \lambda_1 v_1^2 + \dots + \lambda_n v_n^2$, where $\lambda_j > 0$ for all j and, by construction, it is clear that $G = \mathrm{O}(q_0, p_1, \dots, p_r)$. Now, by using Lemma 2.2, we transform $\{q_0, p_1, \dots, p_r\}$ into a pre-realizable family $\mathcal{Q} = \{q_0, \dots, q_{r+1}\}$ such that $G = \mathrm{O}(\mathcal{Q})$, but q_0 already verifies condition (3) from Definition 2.1 so we conclude. \square

Example 2.4. For Σ_n the symmetric group on a finite set of n elements, let M be the permutation $\mathbb{Q}\Sigma_n$ -module with basis $\{m_1, \dots, m_n\}$. Then, given a permutation $\sigma \in \Sigma_n$, the action of σ on M is given by $\sigma(m_i) = m_{\sigma(i)}$ and, if $x_i \in M^*$ denotes the dual of $m_i \in M$, the invariant ring $\mathbb{Q}[M]^{\Sigma_n} \subset \mathbb{Q}[x_1, \dots, x_n]$ is the polynomial algebra generated by the elementary symmetric functions e_i , $i = 1, \dots, n$, defined by the formal equation

$$f(X) = \prod_{i=1}^n (X - x_i) = X^n + \sum_{i=1}^n (-1)^i e_i(x_1, \dots, x_n) X^{n-i}$$

where X is an indeterminate, [4, Example of p. 2].

Now, following the lines of the proof of Theorem 2.3, we obtain that $\Sigma_n = \mathrm{O}(\mathcal{Q}) \leq \mathrm{GL}(M)$ where $\mathcal{Q} = \{q_0, \dots, q_{n+1}\}$ is the following realizable family of n -algebraic forms

$$\begin{aligned} q_0 &= n! \sum_{i=1}^n x_i^2, \\ q_j &= e_j q_{j-1} q_0, \text{ for } j = 1, \dots, n, \\ q_{n+1} &= q_0^s, \text{ for } s \geq \left\lceil \frac{(n+4)(n+1)}{4} \right\rceil + 1. \end{aligned}$$

3. KAHN'S PROBLEM FOR ORTHOGONAL GROUPS

In this section we realize orthogonal groups of a finite family $\mathcal{Q} = \{q_0, \dots, q_r\} \subset \mathbb{Q}[M]$ of rational n -algebraic forms under the hypothesis that $q_0 = \lambda_1 v_1^d + \dots + \lambda_n v_n^d$, for some basis $\{v_1, \dots, v_n\}$ of the dual M^* and some $d > 1$, with $\lambda_i \neq 0$ for $i = 1, \dots, n$.

We recall some results on rational homotopy theory and refer the reader to [10]. If W is a graded rational vector space, we write ΛW for the free commutative graded algebra on W , which is a symmetric algebra on W^{even} tensored with an exterior algebra on W^{odd} . A Sullivan algebra is a commutative differential graded algebra which is free as commutative graded algebra on a simply connected graded vector space W of finite dimension in each degree. It

is minimal if in addition $d(W) \subset \Lambda^{\geq 2}W$. The geometric realization functor of Sullivan, establishes an isomorphism between finite-type minimal Sullivan algebras and simply-connected spaces, and an isomorphism between W and $\pi_*(X)$ as graded modules.

Definition 3.1. Let $\mathcal{Q} = \{q_0, \dots, q_{r+1}\} \subset \mathbb{Q}[v_1, \dots, v_n]$ be a family of realizable n -algebraic forms, where $\{v_1, \dots, v_n\}$ is a basis of M^* and $d = \deg(q_0) \geq 2$. Then, for any k integer satisfying $\deg(q_{r+1}) < 2k + (d - 1)$, we define the minimal Sullivan model

$$\mathcal{M}_{(\mathcal{Q},k)} = \left(\Lambda(x_1, x_2, y_1, y_2, y_3, z, v_j \mid j = 1, \dots, n), d \right)$$

where degrees and differential are described by

$$\begin{aligned} |x_1| &= 8, & d(x_1) &= 0 \\ |x_2| &= 10, & d(x_2) &= 0 \\ |y_1| &= 33, & d(y_1) &= x_1^3 x_2 \\ |y_2| &= 35, & d(y_2) &= x_1^2 x_2^2 \\ |y_3| &= 37, & d(y_3) &= x_1 x_2^3 \\ |v_j| &= 40, & d(v_j) &= 0 \\ |z| &= 80k + 40d - 41, & d(z) &= \sum_{i=1}^{r+1} q_i x_1^{10k+5(d-1)-5 \deg(q_i)} + q_0 (x_1^{10k-5} + x_2^{8k-4}) \\ & & & + x_1^{10k+5(d-4)} (y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6) \\ & & & + x_1^{10k+5(d-1)} + x_2^{8k+4(d-1)}. \end{aligned}$$

We can now prove one of our main results. For clarity purposes, we include some technical lemmas at the end of this paper, in Section 5.

Theorem 3.2. Let $\mathcal{M}_{(\mathcal{Q},k)}$ be the minimal Sullivan model from above. Then, the group of self homotopy equivalences $\mathcal{E}(\mathcal{M}_{(\mathcal{Q},k)})$ is isomorphic to the orthogonal group $O(\mathcal{Q})$.

Proof. For any $g \in O(\mathcal{Q})$, we can define $f_g \in \text{Aut}(\mathcal{M}_{(\mathcal{Q},k)})$ given by:

$$\begin{aligned} f_g(x_i) &= x_i \\ f_g(y_i) &= y_i, \\ f_g(v_j) &= g \cdot v_j \\ f_g(z) &= z \end{aligned}$$

Hence $O(\mathcal{Q}) \leq \text{Aut}(\mathcal{M}_{(\mathcal{Q},k)})$ and also $O(\mathcal{Q}) \leq \mathcal{E}(\mathcal{M}_{(\mathcal{Q},k)})$. Let us now consider an arbitrary $f \in \text{Aut}(\mathcal{M}_{(\mathcal{Q},k)})$. By degrees reasoning, we have:

$$\begin{aligned} f(x_1) &= a_1 x_1 \\ f(x_2) &= a_2 x_2 \\ f(y_i) &= b_i y_i \\ f(v_j) &= A(v_j) + a_1(j) x_1^5 + a_2(j) x_2^4, \text{ where } A \in \text{GL}(M^*) \\ f(z) &= cz + y_1 A_1 + y_2 A_2 + y_3 A_3 + y_1 y_2 y_3 D, \text{ where } A_1, A_2, D \in \mathbb{Q}[x_1, x_2, v_1, \dots, v_n] \end{aligned} \tag{1}$$

Now, using that $f(dy_i) = df(y_i)$, for $i = 1, 2, 3$ we obtain:

$$\begin{aligned} b_1 &= a_1^3 a_2 \\ b_2 &= a_1^2 a_2^2 \\ b_3 &= a_1 a_2^3 \end{aligned} \tag{2}$$

We now compare $f(dz)$ to $df(z)$ which must be equal:

$$\begin{aligned} f(d(z)) &= \sum_{i=1}^{r+1} f(q_i)(a_1 x_1)^{10k+5(d-1)-5 \deg(q_i)} + f(q_0) \left((a_1 x_1)^{10k-5} + (a_2 x_2)^{8k-4} \right) \\ &\quad + (a_1 x_1)^{10k+5(d-4)} \left((b_1 y_1)(b_2 y_2)(a_1 x_1)^4 (a_2 x_2)^2 - (b_1 y_1)(b_3 y_3)(a_1 x_1)^5 (a_2 x_2) \right. \\ &\quad \left. + (b_2 y_2)(b_3 y_3)(a_1 x_1)^6 \right) + (a_1 x_1)^{10k+5(d-1)} + (a_2 x_2)^{8k+4(d-1)} \end{aligned} \tag{3}$$

$$\begin{aligned} df(z) &= c \left[\sum_{i=1}^{r+1} q_i x_1^{10k+5(d-1)-5 \deg(q_i)} + q_0 (x_1^{10k-5} + x_2^{8k-4}) \right. \\ &\quad \left. + x_1^{10k+5(d-4)} (y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6) + x_1^{10k+5(d-1)} + x_2^{8k+4(d-1)} \right] \\ &\quad + x_1^3 x_2 A_1 + x_1^2 x_2^2 A_2 + x_1 x_2^3 A_3 + x_1^3 x_2 y_2 y_3 D - x_1^2 x_2^2 y_1 y_3 D + x_1 x_2^3 y_1 y_2 D \end{aligned} \tag{4}$$

We first observe that $D = 0$. This is easily obtained since in (4) there are $x_1^3 x_2 y_2 y_3 D$ like terms while in (3) no term in $y_2 y_3$ contains x_2 as a factor.

We now claim that $a_2(j) = 0$, for $j = 1, \dots, n$. In (3) the term $f(q_0)(a_2 x_2)^{8k-4}$ contains the expression

$$\lambda_j \sum_{j=1}^n \binom{d}{1} A(v_j) (a_2(j) x_2)^{4(d-1)} (a_2 x_2)^{8k-4}. \tag{5}$$

Using again that $f(d(z))$ and $df(z)$ must be equal and, since in (4) the only possible summands in $A(v_j) x_2^{8k+4(d-2)}$ arise from those which contain A_i , $i = 1, 2, 3$, which also contain x_1 as a factor, we deduce that the expression (5) is zero. This directly implies that

$$\sum_{j=1}^n \lambda_j d a_2^{8k-4} a_2(j)^{4(d-1)} A(v_j) = 0.$$

Now, $\{A(v_j)\}_{j=1}^n$ are linearly independent, since $A \in \text{GL}(M^*)$ and $\{v_j\}_{j=1}^n$ are linearly independent. Therefore $\lambda_j d a_2^{8k-4} a_2(j)^{4(d-1)} = 0$ for every $j = 1, \dots, n$, and since $a_2^{8k-4} \in \mathbb{Q}^*$, $d > 1$, and $\lambda_i \neq 0$, we get that $a_2(j) = 0$, $j = 1, \dots, n$.

Next we prove that $a_1(j) = 0$, for $j = 1, \dots, n$. To that end, we consider monomials in $d(z)$ containing both x_1 with the least possible exponent, and containing *all* the variables v_1, \dots, v_n . Observe that those monomials exclusively come from $q_{r+1} x_1^{10k+5(d-1)-5 \deg(q_{r+1})}$, and their existence is guaranteed since \mathcal{Q} is a realizable family of forms which in particular implies that $q_{r+1} = q_0^s$ for an adequate s . Now, we order them by lexicographic order, that is, $v_1^{d_1} \dots v_n^{d_n} x_1^{10k+5(d-1)-5 \deg(q_{r+1})} > v_1^{e_1} \dots v_n^{e_n} x_1^{10k+5(d-1)-5 \deg(q_{r+1})}$ if $d_1 > e_1$ or if $d_1 = e_1$,

$d_2 > e_2$, and so on. We pick the highest of those monomials in $d(z)$ which, up to some non-zero coefficient, has the form:

$$v_1^{\alpha_1} \cdots v_n^{\alpha_n} x_1^{10k+5(d-1)-5 \deg(q_{r+1})}$$

Therefore, there exists a monomial in (3), up to some coefficient in the variables $A(v_i)$'s,

$$\binom{\alpha_n}{1} a_1(n) A(v_1)^{\alpha_1} \cdots A(v_n)^{\alpha_n-1} x_1^{10k+5d-5 \deg(q_{r+1})}. \quad (6)$$

Due to the exponent of x_1 , the monomial (6) does not appear in (4) (recall that $\deg(q_i) - \deg(q_j) > 1$ for every $i \neq j$). Therefore it must be either zero or it must be cancelled by other monomial in (3). Observe that it cannot be cancelled from monomials coming from q_j , with $j < r+1$, as we mentioned above, so the only possibility for it to be cancelled is to be the image of a monomial $v_1^{\alpha_1} \cdots v_{i_0}^{\alpha_{i_0}+1} \cdots v_n^{\alpha_n-1} x_1^{10k+5(d-1)-5 \deg(q_{r+1})}$. Now, that monomial cannot appear in q_{r+1} because it is strictly bigger than $v_1^{\alpha_1} \cdots v_n^{\alpha_n} x_1^{10k+5(d-1)-5 \deg(q_{r+1})}$, which by hypothesis was the highest monomial in q_{r+1} of that form. Therefore we conclude that (6) is zero, and consequently $a_1(n) = 0$. The same arguments, reordering the v_i 's successively, show that $a_1(j) = 0$, for $j = 1, \dots, n-1$.

We have mentioned before that every A_i -term in (4), $i = 1, 2, 3$, has a $x_1 x_2$ factor but it does not have a $y_1 y_2$ factor, whereas in (3) every summand with a $x_1 x_2$ factor has also a $y_1 y_2$ factor. Therefore,

$$0 = x_1^3 x_2 A_1 + x_1^2 x_2^2 A_2 + x_1 x_2^3 A_3 = d(y_1 A_1 + y_2 A_2 + y_3 A_3)$$

implying that $y_1 A_1 + y_2 A_2 + y_3 A_3 = dm_z$ for some element m_z , by Proposition 5.4.

Observe that from these conclusions above, the third line in (4) is zero. We compare now the second line from (4) to the second and third lines in (3) and we get

$$\begin{aligned} c &= (a_1)^{10n-6} (a_2)^2 b_1 b_2 \\ &= (a_1)^{10n-5} a_2 b_1 b_3 \\ &= (a_1)^{10n-4} b_2 b_3 \\ &= (a_1)^{10n+5} \\ &= (a_2)^{8n+4} \end{aligned} \quad (7)$$

and, using equations from (2), we also get that

$$c = a_1^{10n-1} a_2^5 = a_1^{10n+5} = a_2^{8n+4},$$

so that $a_1^{2n+1} = 1$, and therefore $a_1 = 1 = a_2 = b_1 = b_2 = b_3 = c = 1$.

We finally compare the first line of (3) to the first line of (4). Since the n -algebraic forms $\{q_0, \dots, q_{r+1}\}$ verify $2 < \deg(q_0) < \deg(q_1) < \dots < \deg(q_{r+1})$, it is straightforward that $f(q_i) = q_i$ for every $i = 0, \dots, r+1$. Recall that the polynomial expression q_i comes from the n -algebraic form $q_i \in \mathbb{Q}[M^*]$ and that $\mathcal{M}_{(\mathbb{Q}, k)}^{40} = M^*$. So as $f|_{M^*} = A \in \text{GL}(M^*)$, then $q_i = A(q_i) = q_i \circ A^t$ for every $i = 0, \dots, r+1$, which implies that $A^t \in \text{O}(\mathcal{Q})$.

Gathering altogether, we have proved that there exists one element $g = (A^t)^{-1} \in \text{O}(\mathcal{Q})$ such that $f = f_g + dm_z$. Since f is homotopically equivalent to f_g , we conclude our proof. \square

Corollary 3.3. *Let $O(p_0, \dots, p_r)$ be the orthogonal group of a family of n -algebraic forms over an n -dimensional \mathbb{Q} -module, such that p_0 is diagonalizable over \mathbb{Q} with all the $\lambda_i \neq 0$ and $\deg(p_0) > 1$. Then, $O(p_0, \dots, p_r)$ can be realized by infinitely many non homotopically equivalent (rational) spaces.*

Proof. By hypothesis $q_0 = \lambda_1 v_1^d + \dots + \lambda_n v_n^d$ for a given basis that we fix. By Lemma 2.2, there exists a family of pre-realizable (realizable under the assumptions) n -algebraic forms $\mathcal{Q} = \{q_0, \dots, q_{r+1}\}$ such that $O(\mathcal{Q}) = O(p_0, \dots, p_r)$. Then, by Theorem 3.2 there exists, for every integer k s.t. $\deg(q_{r+1}) < 2k + (\deg q_0 - 1)$, a minimal Sullivan model $\mathcal{M}_{(\mathcal{Q}, k)}$ that realizes $O(\mathcal{Q})$. It is clear that if k is different from k' , then $\mathcal{M}_{(\mathcal{Q}, k)}$ and $\mathcal{M}_{(\mathcal{Q}, k')}$ are non homotopically equivalent. \square

4. LINEAR ALGEBRAIC GROUPS AS ORTHOGONAL GROUPS OF ALGEBRAIC FORMS

This section is a first attempt to give a complete answer to Kahn's realizability problem. Before that, we take the opportunity to recall what we have achieved in the previous sections. In Section 2 we proved that any finitely generated $\mathbb{Q}G$ -faithful module M , provided that G is finite, is the orthogonal group $O(q_0, \dots, q_r) \leq GL(M)$ of a realizable family of algebraic forms over M , Theorem 2.3. Recall that realizable essentially means that q_0 is diagonalizable over \mathbb{Q} with all the coefficients non zero. Within that framework, the theorem above implies that realizing group actions is equivalent to realizing orthogonal groups in the sense of Kahn. Therefore, bringing back techniques from [8], in Section 3 we showed how to realize any orthogonal group of a realizable family of rational algebraic forms, Theorem 3.2.

This latter approach leads to an alternate way of tackling Kahn's realizability problem that we would like to exploit. The first thing to be aware of is that our techniques involve finitely generated Sullivan algebras over \mathbb{Q} , which in particular implies that their group of self homotopy equivalences is a linear algebraic group defined over \mathbb{Q} [17, Theorem 6.1]. Therefore, the following question can be raised:

Question 4.1. *Is every linear algebraic group defined over \mathbb{Q} the orthogonal group of a realizable family of rational algebraic forms?*

We point out that most of the existing literature on linear algebraic groups is developed over \mathbb{C} so we need to introduce some specific notation. Given an n -dimensional \mathbb{Q} -module M , one refers to the group of matrices with entries in \mathbb{C} as the \mathbb{C} -points of $GL(M)$. An algebraic linear group G over \mathbb{Q} is determined by any subgroup of the \mathbb{C} -points of $GL(M)$ defined by polynomial equations in the entries where the coefficients of the polynomials lie in \mathbb{Q} . We denote by $M_{\mathbb{C}} = M \otimes_{\mathbb{Q}} \mathbb{C}$ the complexification of M . A family of rational algebraic forms $\mathcal{Q} = \{q_0, \dots, q_r\} \subset \mathbb{Q}[M]$ gives rise to a family of complex algebraic forms (defined over \mathbb{Q}) that we also denote by $\mathcal{Q} \subset \mathbb{C}[M_{\mathbb{C}}]$; we write $O_{\mathbb{C}}(\mathcal{Q}) = \{f \in GL(M_{\mathbb{C}}) \mid q_i \circ f = q_i \text{ for every } i = 0, \dots, r\}$ for the orthogonal group. Then, $O(\mathcal{Q})$ equals $O_{\mathbb{C}}(\mathcal{Q})(\mathbb{Q})$, the group of rational points of $O_{\mathbb{C}}(\mathcal{Q})$.

The following argument shows that when we consider orthogonal groups of a family of realizable forms, we are indeed considering orthogonal representations in the classical sense (i.e. preserving a quadratic form).

Lemma 4.2. *Let M be an n -dimensional \mathbb{Q} -module and let $\mathcal{Q} = \{q_0, \dots, q_{r+1}\} \subset \mathbb{Q}[M]$ be a family of realizable algebraic n -forms with $d = \deg(q_0) > 1$. If $O_{\mathbb{C}}(\mathcal{Q})$ is infinite, then $d = 2$.*

Proof. Assume $d > 2$ and let $q_0 = \lambda_1 v_1^d + \dots + \lambda_n v_n^d$ for some basis $\{v_1, \dots, v_n\}$ of the dual M^* , where $\lambda_i \neq 0$ for $i = 1, \dots, n$. Let $(q_0)_i$ denote the i -th partial derivative of q_0 , that is, $(q_0)_i = d\lambda_i v_i^{d-1}$. Then $(0, \dots, 0)$ is the only common zero in $M_{\mathbb{C}}$ of the polynomials $(q_0)_1, \dots, (q_0)_n$, and therefore q_0 is nondegenerate in the sense of [16, §1]. Therefore, according to [16, Theorem 2.1] the group $O_{\mathbb{C}}(q_0)$ is finite (recall $d > 2$). Now, since $O_{\mathbb{C}}(\mathcal{Q})$ is infinite and $O_{\mathbb{C}}(\mathcal{Q}) \subset O_{\mathbb{C}}(q_0)$, we get a contradiction. \square

Remark 4.3. *In view of Theorem 2.3 and Lemma 4.2, given an n -dimensional \mathbb{Q} -module M and a family of realizable n -algebraic forms $\mathcal{Q} = \{q_0, \dots, q_{r+1}\} \subset \mathbb{Q}[M]$, there exists a family of realizable n -algebraic forms $\mathcal{P} = \{p_0, \dots, p_{s+1}\} \subset \mathbb{Q}[M]$ such that $\deg(p_0) = 2$ and $O_{\mathbb{C}}(\mathcal{Q}) = O_{\mathbb{C}}(\mathcal{P})$ (and therefore $O(\mathcal{Q}) = O(\mathcal{P})$).*

The question of whether it is possible to describe algebraic linear groups as orthogonal groups of algebraic forms has been addressed by Garibadi-Guralnick [12] when the group is simple; see also [5]. The following result is a restatement of [12, Theorem 6.6], and we sketch here the proof to emphasize that the algebraic forms involved can be chosen to be defined over \mathbb{Q} .

Theorem 4.4. *Let G be a center free simple linear algebraic group defined over \mathbb{Q} which is not C_2 type. Then there exist a \mathbb{Q} -module M of dimension $n = \dim(\text{Lie}(G))$ and a family of realizable n -algebraic forms $\mathcal{Q} = \{q_0, q_1, q_2\} \subset \mathbb{Q}[M]$ such that the following hold*

- i) $\deg(q_0) = 2$,
- ii) $G \subset O_{\mathbb{C}}(\mathcal{Q}) \subset \text{GL}(M_{\mathbb{C}})$, and this representation is equivalent to the adjoint representation $G \subset \text{GL}(\text{Lie}_{\mathbb{C}}(G))$,
- iii) $[O_{\mathbb{C}}(\mathcal{Q}) : G]$ is finite.

Proof. The proof is along the lines in [12, Section 6]. Let \mathcal{R} be a root system underlying $\text{Lie}(G)$, and define $N = \mathcal{P}^{\vee} \otimes \mathbb{Q}$ where \mathcal{P}^{\vee} denotes the dual of the root lattice \mathcal{R}^{\vee} . If W is the Weyl group of G , then W acts on N and $\mathbb{Q}[N]^W$ is a polynomial algebra with generators p_1, \dots, p_l such that the natural map $\mathbb{Q}[N]^W \otimes \mathbb{C} \rightarrow \mathbb{C}[N_{\mathbb{C}}]^W$ is an isomorphism where $\mathbb{C}[N_{\mathbb{C}}]^W$ is the polynomial algebra generated by the images of p_1, \dots, p_l [12, Lemma 6.3]. Moreover, there is a natural map $\mathbb{C}[\text{Lie}_{\mathbb{C}}(G)]^G \rightarrow \mathbb{C}[N_{\mathbb{C}}]^W$ which is an isomorphism, so we can choose f_1, \dots, f_l homogeneous polynomials defined over \mathbb{Q} by pulling back the polynomials p_1, \dots, p_l above, such that they generate $\mathbb{C}[\text{Lie}_{\mathbb{C}}(G)]^G \rightarrow \mathbb{C}[N_{\mathbb{C}}]^W$. We define M to be dual of the \mathbb{Q} -module spanned by the variables of p_i , so $M_{\mathbb{C}} = \text{Lie}_{\mathbb{C}}(G)$ and $\mathbb{C}[M_{\mathbb{C}}] = \mathbb{C}[\text{Lie}_{\mathbb{C}}(G)]$.

Let f_1 be (a multiple of) the quadratic Killing form on $\text{Lie}_{\mathbb{C}}(G)$, and let f be one of the higher degree invariants f_2, \dots, f_l , which is not a power of q_0 (take $f = f_1$ if G is of type A_1). Then, define $q_0 = f_1$, $q_2 = q_0 f$ and q_3 an appropriate power of q_0 according to Definition 2.1.(3) so that $\mathcal{Q} = \{q_0, q_1, q_2\} \subset \mathbb{Q}[M]$ becomes a family of realizable algebraic forms. Now, according to [12, Theorem 6.6] (and the related discussion in case G is A_1 type), G is the connected component of $O_{\mathbb{C}}(f)$, that is, $[O_{\mathbb{C}}(f) : G]$ is finite. Therefore, since $G \subset O_{\mathbb{C}}(\mathcal{Q}) \subset O_{\mathbb{C}}(f)$, then $[O_{\mathbb{C}}(\mathcal{Q}) : G]$ is finite too. \square

Remark 4.5. *Observe that, since the adjoint representation is faithful only if G is center free, that condition can not be avoided from the previous theorem.*

In general, it is not easy to provide an explicit description of the algebraic forms appearing in Theorem 4.4 (see [12, Example 6.7] for $G = E_8$). Nevertheless, it is possible to construct algebraic forms that fit in this setting:

Example 4.6. *Let $G = E_8$. Since $n = \dim(\text{Lie}(E_8)) = 248$, we have to consider algebraic 248-forms. Let f_1 be a G -invariant quadratic algebraic form defined over \mathbb{Q} as in the proof of Theorem 4.4, and let f be the polynomial of degree 8 described in [7, Equation (2.3)]. Then $\mathcal{Q} = \{q_0, q_1, q_2\}$, where $q_0 = f_1$, $q_1 = q_0 f$ and $q_2 = q_0^{248}$, is a realizable family of algebraic forms such that $O_{\mathbb{C}}(\mathcal{Q}) = E_8 \times \mathbb{Z}/2$.*

Indeed, according to [12, Theorem 3.1], $O_{\mathbb{C}}(f) = E_8 \times \mathbb{Z}/8$ where the factor $\mathbb{Z}/8$ is generated by diagonal matrices of the form $\omega \mathbb{I}$ with $\omega^8 = 1$. But since f_1 is a non trivial quadratic polynomial, a diagonal matrix $\omega \mathbb{I}$ stabilizes f_1 , if and only if, $\omega^2 = 1$. Therefore $O_{\mathbb{C}}(\mathcal{Q}) = O_{\mathbb{C}}(f_1, f) = E_8 \times \mathbb{Z}/2$, where the factor $\mathbb{Z}/2$ is generated by diagonal matrices of the form $\omega \mathbb{I}$ with $\omega^2 = 1$.

Combining Theorems 3.2 and 4.4 we obtain the following.

Corollary 4.7. *Let G be a center free simple linear algebraic group defined over \mathbb{Q} which is not C_2 type. Then there exists a rational space X such that $G(\mathbb{Q})$, the group of rational points of G , is a subgroup of $\mathcal{E}(X)$ and $[\mathcal{E}(X) : G(\mathbb{Q})]$ is finite.*

The following example illustrates that apart from the adjoint representation in Theorem 4.4, other representations can be used to describe a simple linear algebraic group as the orthogonal group of a family of realizable algebraic forms, leading to refinements of Corollary 4.7.

Example 4.8. *Let $G = G_2$, and let $M = \mathbb{Q}^{\oplus 7}$. If we fix $\mathbb{Q}[M] = \mathbb{Q}[x_0, x_i, x'_i : i = 1, 2, 3]$ then according to [3, pag. 194] $G_2 = O_{\mathbb{C}}(f_0, f_1)$ where f_0 is the quadratic form*

$$f_0 = -2x_0^2 + x_1x'_1 + x_2x'_2 + x_3x'_3,$$

and f_1 is Dickson alternating trilinear form

$$f_1 = x_0x_1x'_1 + x_0x_2x'_2 + x_0x_3x'_3 + x_1x_2x_3 + x'_1x'_2x'_3.$$

Considering the change of variables $v_1 = x_0$, and $v_{2i} = x_i + x'_i$, $v_{2i+1} = x_i - x'_i$ for $i = 1, 2, 3$, we obtain in $\mathbb{Q}[M] = \mathbb{Q}[v_i : i = 1, \dots, 7]$ the expressions

$$f_1 = \frac{1}{4} \left(v_1(v_2^2 - v_3^2 + v_4^2 - v_5^2 + v_6^2 - v_7^2) + v_2v_5v_7 + v_3v_4v_7 + v_3v_5v_6 + v_2v_4v_6 \right)$$

and

$$f_0 = -2v_1 + \frac{1}{4}v_2^2 - \frac{1}{4}v_3^2 + \frac{1}{4}v_4^2 - \frac{1}{4}v_5^2 + \frac{1}{4}v_6^2 - \frac{1}{4}v_7^2,$$

that give us a family of realizable 7-forms $\mathcal{Q} = \{q_0, q_1, q_2\}$, where $q_0 = f_0$, $q_1 = f_1$, and $q_2 = f_0^7$, such that $G_2 = O_{\mathbb{C}}(\mathcal{Q})$. Therefore, according to Theorem 3.2, there exists a space X such that $\mathcal{E}(X) = O(\mathcal{Q}) = G_2(\mathbb{Q})$.

5. LEMMAS ABOUT ELEMENTS

We collect here technical results on the algebraic structure of the Sullivan model $\mathcal{M}_{(\mathbb{Q},k)}$ introduced in Definition 3.1. They are needed in the proof of Theorem 3.2.

In what follows $P = \mathbb{Q}[x_1, x_2, v_j : j = 1, \dots, n]$.

Lemma 5.1. *Let $A_1 \in P^{80k+40d-74}$, for $d > 1, k > 1$. Then $A_1 = x_1^2 x_2^3 B_1$ with $B_1 \in P$.*

Proof. Let a, b, c be non negative integers such that $x_1^a x_2^b p \in P^{80k+40d-74}$ where $p \in \mathbb{Q}[v_j]$ is an algebraic form of degree c . Then $8a + 10b + 40c = 80k + 40d - 74$ which implies $8a \equiv 6 \pmod{10}$ and hence, $a \equiv 2, 7 \pmod{10}$. Then, since $a \geq 2$, $x_1^a x_2^b p = x_1^2 (x_1^{\tilde{a}} x_2^b p)$ with $(x_1^{\tilde{a}} x_2^b p) \in P^{80k+40d-90}$. Simplifying notation, we write $A_1 = x_1^2 \tilde{A}_1$, where $\tilde{A}_1 \in P^{80k+40d-90}$. Now, let $x_1^{\tilde{a}} x_2^b p \in P^{80k+40d-90}$. Then, $8\tilde{a} + 10b + 40c = 80k + 40d - 90$, which implies $2b \equiv 6 \pmod{8}$ and hence $b \equiv 3, 7 \pmod{8}$. In particular, $b \geq 3$ so $\tilde{A}_1 = x_2^3 B_1$, where $B_1 \in P^{80k+40d-120}$. \square

Lemma 5.2. *Let $A_2 \in P^{80k+40d-76}$, for $d > 1, k > 1$. Then $A_2 = x_1^3 x_2^2 B_2$ with $B_2 \in P$.*

Proof. Let a, b, c be non negative integers such that $x_1^a x_2^b p \in P^{80k+40d-76}$ where $p \in \mathbb{Q}[v_j]$ is an algebraic form of degree c . Then $8a + 10b + 40c = 80k + 40d - 76$ which implies $8a \equiv 4 \pmod{10}$ and hence, $a \equiv 3, 8 \pmod{10}$. Then, since $a \geq 3$, $x_1^a x_2^b p = x_1^3 (x_1^{\tilde{a}} x_2^b p)$ with $(x_1^{\tilde{a}} x_2^b p) \in P^{80k+40d-100}$. Simplifying notation, we write $A_2 = x_1^3 \tilde{A}_2$, where $\tilde{A}_2 \in P^{80k+40d-100}$. Now, let $x_1^{\tilde{a}} x_2^b p \in P^{80k+40d-100}$. Then, $8\tilde{a} + 10b + 40c = 80k + 40d - 100$, which implies $2b \equiv 4 \pmod{8}$ and hence $b \equiv 2, 6 \pmod{8}$. In particular, $b \geq 2$ so $\tilde{A}_2 = x_2^2 B_2$, where $B_2 \in P^{80k+40d-120}$. \square

Lemma 5.3. *Let $A_3 \in P^{80k+40d-78}$, for $d > 1, k > 1$. Then $A_3 = x_1^4 x_2 B_3$ with $B_3 \in P$.*

Proof. Let a, b, c be non negative integers such that $x_1^a x_2^b p \in P^{80k+40d-78}$ where $p \in \mathbb{Q}[v_j]$ is an algebraic form of degree c . Then $8a + 10b + 40c = 80k + 40d - 78$ which implies $8a \equiv 2 \pmod{10}$ and hence, $a \equiv 4, 9 \pmod{10}$. Then, since $a \geq 4$, $x_1^a x_2^b p = x_1^4 (x_1^{\tilde{a}} x_2^b p)$ with $(x_1^{\tilde{a}} x_2^b p) \in P^{80k+40d-110}$. Simplifying notation, we write $A_3 = x_1^4 \tilde{A}_3$, where $\tilde{A}_3 \in P^{80k+40d-110}$. Now, let $x_1^{\tilde{a}} x_2^b p \in P^{80k+40d-110}$. Then, $8\tilde{a} + 10b + 40c = 80k + 40d - 110$, which implies $2b \equiv 2 \pmod{8}$ and hence $b \equiv 1, 5 \pmod{8}$. In particular, $b \geq 1$ so $\tilde{A}_3 = x_2 B_3$, where $B_3 \in P^{80k+40d-120}$. \square

Proposition 5.4. *Let $y_1 A_1 + y_2 A_2 + y_3 A_3 \in \mathcal{M}^{80k+40d-41}$ such that $A_i \in P$ and $d(y_1 A_1 + y_2 A_2 + y_3 A_3) = 0$. Then, there exists one element $m \in \mathcal{M}^{80k+40d-42}$ such that $y_1 A_1 + y_2 A_2 + y_3 A_3 = d(m)$.*

Proof. Using Lemma 5.1, Lemma 5.2 and Lemma 5.3, we know that $A_1 = x_1^2 x_2^3 B_1$, $A_2 = x_1^3 x_2^2 B_2$ and $A_3 = x_1^4 x_2 B_3$. A straightforward computation shows that $d(y_1 A_1 + y_2 A_2 + y_3 A_3) = x_1^5 x_2^4 (B_1 + B_2 + B_3)$. As by hypothesis, $d(y_1 A_1 + y_2 A_2 + y_3 A_3) = 0$, we obtain that $B_1 + B_2 + B_3 = 0$.

Now

$$\begin{aligned}
y_1A_1 + y_2A_2 + y_3A_3 &= \\
&= y_1(x_1^2x_2^3(-B_2 - B_3)) + y_2(x_1^3x_2^2B_2) + y_3(x_1^4x_2B_3) \\
&= x_1^3x_2y_2(x_2B_2) - y_1x_1^2x_2^2(x_2B_2) + x_1^3x_2y_3(x_1B_3) - y_1x_1x_2^3(x_1B_3) \\
&= d(y_1y_2(x_2B_2)) + d(y_1y_3(x_1B_3))
\end{aligned} \tag{8}$$

Hence $m = y_1y_2(x_2B_2) + y_1y_3(x_1B_3)$

□

Acknowledgements. The authors would like to express their gratitude to Professors Skip Garibaldi, Bob Guralnick and Hiroo Shiga for helpful comments and their interest on the subject.

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